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1996 J. Phys. A: Math. Gen. 29 L377

(http://iopscience.iop.org/0305-4470/29/15/002)

J. Phys. A: Math. Gen. **29** (1996) L377–L379. Printed in the UK

LETTER TO THE EDITOR

On the self-fractional Fourier functions

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Received 18 April 1996

Abstract. It is shown that a self-fractional Fourier function for an angle $2\pi N/M$, where N and M are indivisible integers ($N < M$), is also a self-fractional Fourier function for angles $2\pi j/M$ ($j = 1, 2, \ldots$). This allows us to define a self-fractional Fourier function of order M. An eigenvalue of a fractional Fourier operator for angle $2\pi j/M$ is equal to $\exp(\pm i2\pi Lj/M)$, where L is an integer.

Self-imaging phenomena in optical or, in general, in physical systems refer to the fact that an input function (wave field amplitude or wavefunction) is an eigenfunction of the operator describing the given system. In this letter self-imaging in physical systems described by the fractional Fourier transform (fractional FT) [1] is investigated. It is related to the analysis of self-fractional Fourier functions (SFFFs), introduced in [2] , which are their own fractional FTs at some angle, i.e. they are eigenfunctions of the corresponding fractional FT operator with an eigenvalue equal to 1. SFFFs cover, as a particular case, self-Fourier functions (SFFs) [3–6], whose Fourier transforms (FTs) are identical to themselves. It has also been shown [2] how to generate an SFFF for an arbitrary angle from any transformable function.

In this letter the definition of an SFFF is generalized to the case of arbitrary eigenvalues, whose possible values are determined. It is shown that an SFFF for any given angle α is also one for angles $2\pi j/M$, where M is some integer dependent on α and $j = 1, 2, \ldots$. This allows us to define an SFFF of order M and to clarify the procedure for the generation of SFFFs.

The fractional FT at an angle α of a function $f(x)$ is given by [7, 8]

$$\left[R^{\alpha} f(x)\right](u) = \int_{-\infty}^{\infty} f(x) K_{\alpha}(x, u)\, \mathrm{d}x \tag{1}$$

with the kernel

$$K_{\alpha}(x, u) = \sqrt{\frac{1 - \mathrm{i}\cot\alpha}{2\pi}} \exp\left(\mathrm{i}\frac{\cos\alpha(x^2 + u^2) - 2xu}{2\sin\alpha}\right) \tag{2}$$

being, except for a phase shift $\alpha/2$, equal to the propagator of the non-stationary Schrödinger equation for a harmonic oscillator [9]. This equation also describes in a paraxial approximation the wave propagation through a quadratic refractive index medium [8, 10, 11]. If α or $\alpha + \pi$ is a multiple of 2π the kernel $K_{\alpha}(x, u)$ reduces to $\delta(x - u)$ or $\delta(x + u)$ respectively. Thus, the fractional FT at angle $2\pi n$ (n is integer) corresponds to the identity operator. For $\alpha = \pi/2$ relationship (1) is the ordinary Fourier transform (FT).

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0305-4470/96/150377+03$19.50

We define here SFFFs in a slightly generalized sense as has been done in [4] for the case of SFFs. A function is a self-fractional Fourier function for angle α if it satisfies the following equation:

$$R^{\alpha}\left[f_{\alpha}(x)\right](u) = Af_{\alpha}(u) \tag{3}$$

where A is a complex constant factor. In other words, $f_{\alpha}(x)$ is an eigenfunction of the corresponding fractional FT operator with eigenvalue A. From (1), (2) and (3) it follows that any arbitrary function is an SFFF for $\alpha = 2\pi n$, a symmetric (even or odd) function is an SFFF for $\alpha = \pi n$ and an SFF is an SFFF for $\alpha = \pi n/2$, where n is an integer.

The fractional FT is the periodic transform

$$R^{\alpha+2\pi n}\left[f(x)\right](u) = R^{\alpha}\left[f(x)\right](u). \tag{4}$$

Therefore, one can always represent any angle α in the form $\alpha = 2\pi N/M$, where N and M are indivisible integers and $N < M$.

Let us now prove that an SFFF for any angle $\alpha = 2\pi N/M$ is also one for the angle $2\pi/M$ and *vice versa*. From the additive property [7]

$$R^{\alpha}\left[R^{\beta}\left[f(x)\right](v)\right](u) = R^{\alpha+\beta}\left[f(x)\right](u) \tag{5}$$

and (3) it immediately follows that if a function is an SFFF for $\alpha = 2\pi N/M$ it is also one for $\alpha_k = 2\pi kN/M$ $(k = 1, 2, \ldots)$

$$\begin{aligned} R^{2\pi kN/M}\left[f(x)_{2\pi N/M}\right](u) &= R^{2\pi(k-1)N/M}\left[R^{2\pi kN/M}\left[f(x)_{2\pi N/M}\right](v)\right](u) \\ &= AR^{2\pi(k-1)N/M}\left[f(x)_{2\pi N/M}\right](u) = \cdots = A^{k}f(x)_{2\pi N/M}. \end{aligned} \tag{6}$$

In the particular case $N = 1$ we obtain that if a function is an SFFF for angle $2\pi/M$ then it is also one for the angles $2\pi k/M$, where k is an integer.

Taking into account the periodic property (4) let us show that angle $\alpha_k = 2\pi kN/M$ for some k reduces to angle $2\pi/M$ and, therefore, an SFFF for $\alpha = 2\pi N/M$ is an SFFF for angle $2\pi/M$. The fractional FT is cyclic, which means that after applying the cascade of L fractional FTs at the angle α one retrieves an input function

$$R^{L\alpha}\left[f(x)\right](u) = f(u) \tag{7}$$

where the integer L depends on the angle α. Note that for $\alpha = 2\pi N/M$ the smallest integer L satisfying (7) is equal to M, because N and M are indivisible integers and $N < M$. By using the periodic property (4) any intermediate angle from this cascade $2\pi kN/M$ $(k = 1, \ldots, M)$ can be represented in the form $2\pi j/M$, where j is some integer in the region $[1, \ldots, M]$. Moreover, all integers j are different, because under the opposite assumption one has the smallest integer $L < M$. So, one cycle of the fractional FTs at angle $\alpha = 2\pi N/M$ consists of M fractional FTs at the different angles $2\pi j/M$ $(j = 1, \ldots, M)$ inside one period.

Therefore, if a function is an SFFF for an angle $\alpha = 2\pi N/M$, where N and M are indivisible integers ($N < M$), it is also an SFFF for angle $2\pi/M$ and *vice versa*. As follows from (7), this function is also an SFFF for angles $2\pi j/M$ $(j = 1, 2, \ldots)$. This fact allows us to define an SFFF of order M, which is an SFFF for angles $2\pi j/M$. Note that an SFFF of order Mm, where m is an integer, is also an SFFF of order M. Consequently, an SFFF for angle $\pi/(2M)$ is also an SFF. So, during one period of the fractional FT, self-imaging of an SFFF of order M is observed no less than M times.

Let us now define the eigenvalues of fractional FT operators. From the Parserval relation for the fractional FT [7]

$$\int_{-\infty}^{\infty} |f_{\alpha}(u)|^2\,\mathrm{d}u = \int_{-\infty}^{\infty} \left|R^{\alpha}\left[f_{\alpha}(x)\right](u)\right|^2 \mathrm{d}u = |A|^2 \int_{-\infty}^{\infty} |f_{\alpha}(u)|^2\,\mathrm{d}u \tag{8}$$

it follows that $|A| = 1$. Therefore, a factor A can be represented in the form $A = \exp(\pm \mathrm{i}2\pi\varphi)$, where φ is a real constant. If φ is an integer, then $A = 1$ and one has the exact self-reproducing.

Let us consider an SFFF of order M with some eigenvalue $A = \exp(\pm \mathrm{i}2\pi\varphi)$. From the cyclic property of the fractional FT (4) it follows that the cascade of M fractional FTs at angle $2\pi/M$ yields to self-reproducing itself, then $A^M = \exp(\pm \mathrm{i}2\pi\varphi M) = 1$. This means that $\varphi = L/M$, where L is an integer. Therefore, an eigenvalue A of the fractional FT operator for angle $\alpha = 2\pi/M$ is equal to $A = \exp(\pm \mathrm{i}2\pi L/M)$.

Finally, let us generate an SFFF for an angle α with some factor A.

As has been shown in [2] an SFFF with $A = 1$ for angle $\alpha = 2\pi N/M$ can be constructed from a generating function $g(x)$ through the operation

$$f(x)_{2\pi N/M} = \left[R^0 + R^{2\pi N/M} + R^{2\pi 2N/M} + \cdots + R^{2\pi(k-1)N/M}\right]\left[g(u)\right](x) \tag{9}$$

where k and L are the smallest integers that satisfy $kN/M = L$.

In accordance with the above analysis an SFFF for any angle $\alpha = 2\pi N/M$ is an SFFF of order M. Note that an SFFF of order M can be an SFFF of order Mm. Therefore, an SFFF for an angle $\alpha = 2\pi N/M$ is, in general, an SFFF of order $J = Mm$, which is generated from an arbitrary transformable function $g(x)$ through the operation

$$f(x)_{2\pi/J} = \sum_{k=1}^{J} \exp\left(\mp\frac{\mathrm{i}2\pi Lk}{J}\right) R^{2\pi(k-1)/J}\left[g(u)\right](x) \tag{10}$$

where integer $m \geqslant 1$. The smallest number of terms in sum (10) is M. Indeed, by using the property (4) one obtains that

$$\begin{aligned} R^{2\pi/J}\left[f(x)_{2\pi/J}\right](u) &= \sum_{k=1}^{J} \exp\left(\mp\frac{\mathrm{i}2\pi Lk}{J}\right) R^{2\pi/J}\left[R^{2\pi(k-1)/J}\left[g(v)\right](x)\right](u) \\ &= \sum_{j=1}^{J} \exp\left(\mp\frac{\mathrm{i}2\pi L(j-1)}{J}\right) R^{2\pi(j-1)/J}\left[g(x)\right](u) \\ &= \exp\left(\pm\frac{\mathrm{i}2\pi L}{J}\right) f(u)_{2\pi/J}. \end{aligned} \tag{11}$$

The corresponding eigenvalue for the fractional FT operator for angle $\alpha = 2\pi j/J$ $(j = 1, \ldots, J)$ is equal to $\exp(\pm \mathrm{i}2\pi Lj/J)$, therefore for angle $\alpha = 2\pi N/M$ $(j = mN)$ $A = \exp(\pm \mathrm{i}2\pi LN/M)$.

The author would like to acknowledge support from the Universidad Autonoma de Madrid.

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